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ORIGINAL ARTICLE

Sets and Functions in Theoretical Physics

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Abstract It is easy to show that in many natural axiomatic formulations of physical and even mathematical theories, there are many superfluous concepts usually assumed as primitive. This happens mainly when these theories are formulated in the language of standard set theories, such as Zermelo–Fraenkel's. In 1925, John von Neumann created a set theory where sets are definable by means of functions. We provide a reformulation of von Neumann's set theory and show that it can be used to formulate physical and mathematical theories with a lower number of primitive concepts very naturally. Our basic proposal is to offer a new kind of set-theoretic language that offers advantages with respect to the standard approaches, since it doesn't introduce dispensable primitive concepts. We show how the proposal works by considering significant physical theories, such as non-relativistic classical particle mechanics and classical field theories, as well as a well-known mathematical theory, namely, group theory. This is a first step of a research program we intend to pursue.

1 Introduction

Loosely speaking, in an axiomatic system *S*, a primitive term or concept *c* is definable by means of the remaining primitive concepts if and only if there is an appropriate formula, provable in the system, that "fixes the meaning" of *c* as a function of the other primitive terms of *S*. When *c* is not definable in *S*, it is said to be independent of the other primitive terms. Every definable concept is eliminable,

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in the sense that the *definiendum* can be replaced by the *definiens* in either a given context or in an explicit way.

There is a method, roughly introduced by Padoa (1900) and further developed by other logicians, which can be employed to show either the independence or the dependence of concepts with respect to the remaining concepts (primitive and previously defined). In fact, Padoa's method gives a necessary and sufficient condition for independence of concepts in many formal situations.

In order to present Padoa's method, some preliminary remarks are necessary. If we are working in set theory as our basic framework, an axiomatic system S is characterized by a species of structures (da Costa and Chuaqui 1988). Actually, there is a close relation between species of structures and Suppes predicates (2002) (for details see da Costa and Chuaqui 1988). However, if our underlying logic is a higher-order logic (type theory), S determines a usual higher-order structure (Carnap 1958). In the first case, our language is the first-order language of set theory, and, in the second, it is the language of some type theory. Tarski (1983) showed that Padoa's method is valid in the second case, and Beth (1953) that it is applicable in the first. A simplified and sufficiently rigorous formulation of Padoa's method, adapted to our exposition from (Suppes 1957), can now be described.

Let S be an axiomatic system whose primitive concepts are $c_1, c_2, \ldots, c_n$. One of these concepts, say c_i, is independent (undefinable) from the remaining if and only if there are two models of S in which $c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n$ have the same interpretation, but the interpretations of c_i in such models are different. (Of course, a model of S is a set-theoretic structure in which all axioms of S are true, according to the interpretation of its primitive terms (Mendelson 1997).)

As an example, consider a very simple axiomatic system, namely, a *minimalist space* $\langle X, f \rangle$, whose axioms are:

MS1 X is a non-empty set.
MS2 f is a function whose domain and codomain are both X.

By using Padoa's method we can easily prove, e.g., that f is undefinable, since we can exhibit two models of a minimalist system such that X has the same interpretation in both models but f has two different interpretations within these models. Consider, for example, the Model A, where X is interpreted as the set of real numbers $\Re$ and f is the identity function $f(x) = x$ defined on X; and the Model B, where X is interpreted again as the set $\Re$, but f is the function $f(x) = 2x$, with the same domain X. This means that the interpretation of X does not fix the interpretation of f. In other words, f cannot be defined (or fixed) from X. However, X *is* definable, since any two models with two different interpretations for X would unavoidably entail different interpretations for f. The reason for this is based on the fact that the domain and the codomain of a function f are ingredients of the function itself, at least within the scope of a standard set theory like Zermelo–Fraenkel's. Different domains imply different functions.

So, at least two questions remain:

1. How to define X?
2. What does it mean to say that X is eliminable?

The answers are:

1. $X = dom(f) = cod(f)$ (X is the domain and the codomain of f).
2. We do not need to explicitly mention X. We could rephrase the definition of a minimalist system by saying that a minimalist system is just a function f whose domain is equal to its codomain.

In a similar way, it is possible to prove that in usual axiomatic frameworks for physical theories, time and spacetime are concepts that are definable, and so, eliminable. This happens because time and spacetime are usually considered as domains of functions that describe forces, fields, currents, and so on. For example, according to Padoa's principle, the primitive concept time (described as an interval of real numbers) in a physical theory is independent from the remaining primitive concepts (mass, position, force, speed, magnetic field etc.) if, and only if, there are two models of the physical theory such that time has two interpretations and the remaining primitive symbols have the same interpretation. But usually these two interpretations are not possible, since mass, position, force, speed, magnetic field and other physical concepts are in general described as functions whose domains are time. If we change the interpretation of time, we change the interpretation of the other primitive concepts. So, time is not independent and hence can be defined. Since time is definable, it is eliminable. Time is eliminable in the sense that many physical theories can be rewritten without any explicit mention of time. A similar argument can be used to dispense with spacetime. (Details about this approach can be found in da Costa and Sant'Anna 2001, 2002.)

Results of this kind suggest the idea that functions are indispensable, but the explicit presence of sets as the domains of these functions is questionable. From an intuitive point of view, our results may shed some new light on the explanatory role of sets in theoretical physics, especially at a foundational level. Sets seem to be carried along as just a surplus structure of the mathematical framework in which these theories are formulated. Moreover, in the context of standard set theories, such as Zermelo–Fraenkel's, to reformulate a physical theory without any explicit mention of either time or spacetime is not an easy task—after all, the latter notions are typically expressed in terms of sets. Such reformulations of physical theories in Zermelo–Fraenkel are also unnatural, given that usually the functions that are invoked in the theories demand an explicit mention of their domains, and in this way, sets are brought back. (For an example of a mathematical description of thermodynamics without any explicit mention of time, see da Costa and Sant'Anna 2002.)

What would happen if we could avoid any explicit mention of domains of functions? Could we obtain better axiomatic formulations of physical theories? Could we avoid the presence of time and spacetime structures in a natural way? Could we go more directly to the point, i.e., to the functions that usually describe fields and forces, tensors and metrics, speeds and accelerations?

It may be thought that category theory provides a framework to develop this sort of approach. After all, category theory deals primarily with "functions", called morphisms (see Mac Lane 1994). However, even morphisms have domains, which are other morphisms, and so we still wouldn't have the appropriate framework to develop the approach we have in mind.

What we are looking for is a mathematical theory where functions have no domains at all. In this way, we would immediately avoid the introduction of superfluous primitive notions, such as sets or domains, when we use this theory as the mathematical basis for the formulation of physical theories.

There's also an additional reason to take functions as primitive rather than sets: the notion of a function seems to be more basic. Sets, in an intuitive sense, are the result of a process of collecting objects. An object is collected if it is assigned to a given set. But more fundamental than to collect something is to be able to attribute something to a certain collection. After all, to collect an object means to attribute it to a given collection. But this notion of attributing something to a given collection looks like a function.

From another point of view (closer to what is done in formal theories), we should recall that sets and functions are meant to correspond to an intuitive notion of properties. Usually properties allow to define either classes or sets (see, for example, the Separation Schema in Zermelo–Fraenkel set theory, in the next section). But another possibility is that properties correspond to functions. Talking about objects that have a given property P corresponds to associate certain objects to a label that represents P; and any other remaining objects are supposed to be associated to a different label. The correspondence itself between P and a given label does have a functional, rather than a set-theoretical, appeal. And usually, those labels are called sets. So, why do we need sets? Why can't we deal only with functions? In other words, why can't we label those intended properties with functions instead of sets?

In 1925, John von Neumann (1967) introduced his axiomatization of set theory. There are two major assumptions in his approach, namely, the use of two kinds of collections, sets and classes, and the use of functions as the intuitive basic notion, instead of sets or classes. More specifically, von Neumann deals with three kinds of terms: I-objects (arguments), II-objects (characteristic functions of classes), and I-II-objects (characteristic functions of sets). The axiomatic system originally proposed was further developed by R. M. Robinson, P. Bernays, and Kurt Gödel, and it came to be known as the von Neumann–Bernays–Gödel (NBG) set theory. However, NBG is not faithful to the idea of the priority of functions instead of collections. In the end, NBG is a standard approach to set theory, where the novelty is the use of classes (mainly proper classes: those classes which are not sets) besides sets.

In this paper, we modify von Neumann's set theory in a way quite different from the NBG proposal. In fact, our approach is quite different even from the original proposal of von Neumann's. We wont focus on the use of classes, since classes do not seem to be very useful for physicists interested in basic applications. (In any case, such classes are typically relevant to deal with set-theoretic paradoxes.) But we keep the idea of considering functions as basic objects. At the same time, we show that this new theory, called $\mathcal{N}$, can be used to formulate all standard mathematics as well as set-theoretical axiomatic systems for physical theories.

Von Neumann was very clear about his commitment to the priority of functions in his formulation of set theory. He was perfectly aware that in the usual axiomatizations of set theory, particularly the one proposed by Zermelo and Fraenkel, the notion of set was taken as primitive, and functions were defined in terms of sets. However, von Neumann carefully considered the role that functions play even in those axiomatizations, and noted, in particular, the fact that the central axioms of these set theories invoke the notion of function. As a result, instead of starting with sets and obtain functions from them, it made perfect sense to introduce functions directly as a primitive notion, bypassing altogether the need for assuming sets as primitive. To provide a formally simpler characterization of set theory—one that was, in particular, finitely axiomatizable even using a first-order language—was a major goal for von Neumann. As he points out (von Neumann 1967):

> The reason for this departure from the usual way of proceeding [avoiding sets as primitive notions] is that every axiomatization of set theory uses the notion of function (axiom of separation, axiom of replacement, [...]), and thus it is formally simpler to base the notion of set on that of function than conversely.

Roughly speaking, we take the notion of a "formally simple axiomatization" to stand for an axiomatization of a theory in which, as much as possible, only indispensable primitive notions are introduced. Of course, this is just an intuitive notion. But it highlights a significant point: a formally simple axiomatization in principle allows for more direct formulations of the relevant theories, in the sense that it requires less concepts that are definable (eliminable). In particular, by dispensing with unnecessary primitive notions, a formally simple axiomatization is not cluttered with a baggage of redundant commitments that can make it more difficult to analyze the explanatory role of the theory's components, especially when we consider the foundations of physical theories. This is significant when we consider that some mathematical terms in a physical theory often are not used in explanations. One simple example is the use of real numbers. No physical measure can really correspond to a real number, since any real number can be represented by an infinite sequence of digits. The best physical measures can deal only with a finite number of decimal places. This elementary example helps us see how difficult it is to analyze the epistemological character of physical theories.

Now, if these mathematical terms end up playing some role in the physical theory—that is, if they are assigned a physical interpretation that turns out to be empirically supported—then these terms will definitely be included in the axiomatization. Otherwise, we are better off simply dispensing with them, whenever this is possible.

It is important to note that the same motivation of finding formally simple axiomatizations that von Neumann had in set theory also applies to the foundations of physics. After all, just as in the case of set theory, physical theories are primarily formulated in terms of functions rather than sets. For example, functions figure centrally in the formulation of the main axioms of standard physical theories. Moreover, differential equations play a crucial role in physical theories, and their solutions are, again, functions. Some few examples are the functions that describe force, linear momentum, position, speed, torque and energy in classical mechanics (Goldstein 1980), the functions that correspond to electromagnetic field, charge and current density and force in classical electrodynamics (Jackson 1975), and the operators (functions) that describe position, time evolution, spin, and momentum in quantum theories (Haag 1992).

So, it is no surprise that someone who has always been engaged with the foundations of physics, as von Neumann was, would insist on the priority of functions. We aim to extend this goal of von Neumann's, by trying to simplify the *use* of set theory in mathematics and physics.

We should point out, however, that there are at least two significant differences between von Neumann's proposal and ours. One is the fact that in von Neumann's theory, there are two kinds of terms: functions and arguments. The universes of arguments and functions can overlap, but there are objects that are only arguments and there terms that are just functions. In $\mathcal{N}$, the set theory that we will provide inspired in von Neumann's work, there is only one kind of term: functions. Furthermore, in von Neumann's set theory there is a binary functional letter that produces ordered pairs. In our approach, this manoeuvre is unnecessary.

2 ZFC Set Theory

Let's start by presenting the standard approach to Zermelo–Fraenkel set theory with the axiom of choice (ZFC). There are two reasons for referring to ZFC at this point. First, by presenting the theory, we can compare our proposal to the standard formulation of set theory. Second, having ZFC in place will be useful for our proof that we can still use standard mathematical results when we adopt $\mathcal{N}$. After all, as we'll show shortly, there's a translation from the language of ZFC into $\mathcal{N}$ such that every translated axiom of ZFC is a theorem in $\mathcal{N}$. However, as we will see, to adopt $\mathcal{N}$ has the significant advantage of providing a whole new universe to work with.

ZFC is a first-order theory with identity and with one predicate letter f_1^2, such that the formula $f_1^2(x,y)$ is abbreviated as $x \in y$, if x and y are terms, and is read as "x belongs to y" or "x is an element of y". The negation $\neg(x \in y)$ is abbreviated as $x \notin y$.

The axioms of ZFC are the following:

ZF1—Extensionality $\forall x \forall y(\forall z(z \in x \Leftrightarrow z \in y) \Rightarrow x = y)$
ZF2—Empty set $\exists x \forall y(\neg(y \in x))$
ZF3—Pair $\forall x \forall y \exists z \forall t(t \in z \Leftrightarrow t = x \vee t = y)$

The pair z is denoted by $\{x, y\}$ if $x \neq y$. Otherwise, $z = \{x\} = \{y\}$.

Definition 1 $x \subseteq y =_{def} \forall z(z \in x \Rightarrow z \in y)$

ZF4—Power set $\forall x \exists y \forall z(z \in y \Leftrightarrow z \subseteq x)$

If $F(x)$ is a formula in ZFC, such that there are no free occurrences of the variable y, then the next formula is an axiom of ZFC:

ZF5$_F$—Separation schema $\forall z \exists y \forall x(x \in y \Leftrightarrow x \in z \wedge F(x))$

The set y is denoted by $\{x \in z / F(x)\}$.

If $\alpha(x, y)$ is a formula where all occurrences of x and y are free, then the following is an axiom of ZFC:

ZF6$_\alpha$—Replacement schema

$$\forall x \exists ! y \alpha(x,y) \Rightarrow \forall z \exists w \forall t(t \in w \Leftrightarrow \exists s(s \in z \wedge \alpha(s,t)))$$

ZF7—Union set $\forall x \exists y \forall z(z \in y \Leftrightarrow \exists t(z \in t \wedge t \in x))$

The set y from **ZF7** is abbreviated as

$$y = \bigcup_{t \in x} t$$

The intersection among sets is defined by using the separation schema as follows:

$$\bigcap_{t \in x} t =_{def} \left\{ z \in \bigcup_{t} t / \forall t(t \in x \Rightarrow z \in t) \right\}$$

ZF8—Infinite $\exists x(\emptyset \in x \wedge \forall y(y \in x \Rightarrow y \cup \{y\} \in x))$

ZF9—Choice $\forall x(\forall y \forall z((y \in x \wedge z \in x \wedge y \neq z) \Rightarrow (y \neq \emptyset \wedge y \cap z = \emptyset)) \Rightarrow \exists y \forall z (z \in x \Rightarrow \exists w(y \cap z = \{w\})))$

As is well known, most if not all classical mathematics can be reformulated in ZFC. As a result, ZFC provides a rich framework for the formulation of physical theories—although perhaps not the most economical. As an alternative, we will now consider a different version of set theory, and explore its use in the foundations of physics.

3 $\mathcal{N}$ Theory

The theory $\mathcal{N}$ is a first-order theory with identity, where the formula $x = y$ should be read as "x is equal to y". The formula $\neg(x = y)$ is abbreviated as $x \neq y$. $\mathcal{N}$ has two constants, namely, $\underline{0}$ and $\underline{1}$. It also has one functional letter $f_1^2(f, x)$, where f and x are terms. If $y = f_1^2(f, x)$, we abbreviate this by $f(x) = y$, and we say that y is the *image* of x by f. All terms of $\mathcal{N}$ are called *functions*. The axioms of $\mathcal{N}$ are the following:

N1—Constants $\neg(\underline{0} = \underline{1})$.

In other words, the constants $\underline{0}$ and $\underline{1}$ are different.

Definition 2 $x \in f$ *iff* $f(x) = \underline{1}$

This last definition can be seen as a kind of generalization of the concept of characteristic function. In ZFC, characteristic functions are such that if $f(x) \neq 1$, then $f(x) = 0$. In $\mathcal{N}$, that does not happen. If $f(x) \neq \underline{1}$, we cannot guarantee that $f(x) = \underline{0}$. If $x \in f$, then we say that x belongs to f. We can also say that x is an element of f. We use $x \notin f$ as an abbreviation of $\neg(x \in f)$.

We show in Sect. 5. that the predicate letter $\in$ can be easily translated as the membership relation $\in$ of standard ZFC set theory.

We can still generalize the membership relation $\in$ as follows:

Definition 3 $x \in_w f$ *iff* $f(x) = w$

In this sense, $x \in f$ is equivalent to $x \in_{\underline{1}} f$. We believe that the predicate letter $\in_w$ can be used to generalize fuzzy set theory, since we are generalizing the notion of standard characteristic function. But that is a task for another paper.

Definition 4 $S(f)$ *iff* $\forall x(f(x) = \underline{1} \vee f(x) = \underline{0})$

We read $S(f)$ either as "the function f is a set" or "f is a set". We will write $\forall_S$ and $\exists_S$ as an abbreviation for the universal and existential quantifiers restricted to the predicate S. In other words, any formula $\forall_S x(A)$ in $\mathcal{N}$ should be read as $\forall x(S(x) \Rightarrow A)$, where A is a formula and S is the predicate given by Definition 4. Any formula $\exists_S x(A)$ is supposed to be read as $\exists x(S(x) \wedge A)$.

N2—Correspondence $\forall f \exists s \forall x((f(x) = \underline{1} \Rightarrow s(x) = \underline{1}) \wedge (f(x) \neq \underline{1} \Rightarrow s(x) = \underline{0}))$

In other words, for every function there is a corresponding set. We denote the set s as $s = \mathcal{S}(f)$ and read this as "s is the *setification* of f".

N3—Extensionality $\forall f \forall g(\forall x(f(x) = g(x)) \Rightarrow f = g))$

Axiom **N3** states that if two functions have the same images, then they are the same. In other words, a function is defined by its images.

Theorem 1 $\forall f \forall g(f = g \Rightarrow \forall x\ ((f(x) = g(x)))$

The proof is obtained by using the substitutivity of identity in the formula $f(x) = f(x)$, which is a theorem in any first-order theory with identity, if $f(x)$ is a term. Since $f = g$, then $f(x) = g(x)$.

N4—Empty function $\exists f \forall x(f(x) \neq \underline{1})$

This last axiom guarantees the existence of a function f such that no other function belongs to it (see Definition 2). Such a function is called an *empty function*. Given the axiom **N2**, for every function there is a corresponding set. In this case, for any empty function there is an *empty set* f such that $\forall x(f(x) = \underline{0})$. When the empty function is a set we denote it by $\emptyset$.

Theorem 2 *The empty set is unique.*

Proof Suppose that there is another empty set $\emptyset'$. If $\emptyset'$ is a set, then its images are supposed to be either $\underline{0}$ or $\underline{1}$. However, $\emptyset'$ cannot have $\underline{1}$ as an image, because it is empty. Then, $\forall x(\emptyset'(x) = \underline{0})$. Since $\forall x(\emptyset(x) = \underline{0})$, then $\forall x(\emptyset(x) = \emptyset'(x))$, which, according to axiom **N3**, entails that $\emptyset' = \emptyset$.

N5—Pair $\forall x \forall y \exists f \forall t(f(t) = \underline{1} \Leftrightarrow t = x \vee t = y)$ □

Any function f that satisfies axiom **N5** is denoted by $f = \{x, y\}$, which is called a *pair*. Intuitively speaking, f may have infinite images, but only arguments x and

y have images equal to $\underline{1}$. If $x = y$, the pair is called a *singleton* and is denoted by either $\{x\}$ or $\{y\}$. Ordered pair is defined as:

$$(x, y) =_{def} \{\{x\}, \{x, y\}\}$$

An ordered n-tuple $(x_1, x_2, \ldots, x_n)$ can be defined analogously. It can be denoted by $\langle x_1, x_2, \ldots, x_n \rangle$.

Definition 5 $f \equiv g$ *iff* $\forall x(f(x) = \underline{1} \Leftrightarrow g(x) = \underline{1})$

Theorem 3

1. $\forall f(f \equiv f)$
2. $\forall f \forall g(f \equiv g \Rightarrow g \equiv f)$
3. $\forall f \forall g \forall h((f \equiv g \wedge g \equiv h) \Rightarrow f \equiv h)$

The proof of this theorem is straightforward. We read $f \equiv g$ as "f is equivalent to g". The negation $\neg(f \equiv g)$ is abbreviated as $f \not\equiv g$.

Theorem 4 *If* $f \equiv g$, *then* $\mathcal{S}(f) = \mathcal{S}(g)$.

The proof is immediate from the definition of setification and from the axiom of extensionality.

Definition 6 $f \subseteq g$ *iff* $\forall x(f(x) = \underline{1} \Rightarrow g(x) = \underline{1})$

Definition 7 $f \subset g$ *iff* $f \subseteq g \wedge f \not\equiv g$

N6—Power function $\forall f \exists g \forall x(g(x) = \underline{1} \Leftrightarrow x \subseteq f)$

The function g is called a *power function* of f. If f is a set, then a corresponding power function g is not necessarily unique. But according to axiom **N2** there is a corresponding power set. This power set of the set f is unique and is denoted by $\mathcal{P}(f)$. The proof of uniqueness is analogous to the proof of the uniqueness of the empty set.

Theorem 5 *The empty set* $\emptyset$ *is subset of any set* x.

The proof is straightforward.

Definition 8 $C(f)$ *iff* $\exists y \forall x\ (f(x) = y)$.

We read $C(f)$ as "f is a constant function". One obvious example of a constant function is the empty set, since $\emptyset(x) = \underline{0}$ for any x.

N7$_F$—Separation schema If $F(x)$ is a formula where all occurrences of x are free and such that g does not occur in $F(x)$, then the following formula is an axiom:

$$\forall_Z f \exists g \forall x(g(x) = \underline{1} \Leftrightarrow f(x) = \underline{1} \wedge F(x))$$

where Z is a unary predicate recursively defined in $\mathcal{N}$ as follows:

Definition 9 If f and x are functions, then:

(i) $Z(f) \Rightarrow S(f)$
(ii) $(Z(f) \wedge f(x) = \underline{1}) \Rightarrow Z(x)$
(iii) $(Z(f) \wedge \neg Z(x)) \Rightarrow f(x) = \underline{0}$

Roughly speaking, $Z(f)$ means that f is a function that behaves like a set in the Zermelo–Fraenkel sense. We read $Z(f)$ as "f is a ZF-set". Such a terminology is used because there is a copy of ZFC in $\mathcal{N}$ theory. The meaning of this claim is clarified in Sect. 5, where we prove that the standard mathematics represented by ZF set theory is preserved in $\mathcal{N}$. In other words, the reason why the first quantifier is restricted to Z is to allow us to run the Proof of Lemma 5 in Sect. 5. As noted above, one of our constraints is to preserve classical mathematics in $\mathcal{N}$.

In other words, the separation schema works only for ZF-sets. The reason for such a restriction is to avoid some paradoxes, like the one entailed by the existence of a function g such that $g(x) = \underline{0} \Leftrightarrow \emptyset(x) = \underline{0} \wedge x(x) \neq \underline{0}$ (if the universal quantifier was unrestricted). Since $\emptyset(x) = \underline{0}$ for any x, then we have a function g such that $g(g) = \underline{0}$ and $g(g) \neq \underline{0}$, which is some sort of $\mathcal{N}$ version of Russell's Paradox.

N8$_\alpha$—Replacement schema If $\alpha(x, y)$ is a formula, where all the occurrences of x and y are free, then the following formula is an axiom

$$\forall x \exists !_Z y(\alpha(x,y)) \Rightarrow (\forall f \exists g \forall y(g(y) = \underline{1}) \Leftrightarrow \exists_Z x(f(x) = \underline{1} \wedge \alpha(x,y)))$$

N9—Union of functions $\forall f \exists g \forall x(g(x) = \underline{1} \Leftrightarrow \exists_Z y(y(x) = \underline{1} \wedge f(y) = \underline{1}))$

Just as above, the last quantifier is restricted so that we can run the Proof of Lemma 7 in the next section. The function g is called the *union of sets y* that belong to the function f. If g is a set, then there is no other set that is the union of sets y that belong to f. In this case, we denote g by:

$$\bigcup_{y \in f} y$$

If there are only two functions a and b that belong to y, and if g is a set, then we denote the union g as $a \cup b$.

N10—Infinite function

$$\exists i(i(\emptyset) = \underline{1} \wedge \forall x(i(x) = \underline{1} \Rightarrow \forall y(x \subset y \Rightarrow i(y) = \underline{1})))$$

The function i is called "infinite" because there are infinitely many functions that belong to it. The word "infinite" here has still an intuitive meaning.

N11—Choice

$$\begin{aligned}&\forall_Z f(\forall x \forall y((f(x) = \underline{1} \wedge f(y) = \underline{1} \wedge x \neq y) \Rightarrow \\ &(x \neq \emptyset \wedge \neg \exists s(x(s) = y(s) = \underline{1}))) \Rightarrow \\ &\exists t \forall r(f(r) = \underline{1} \Rightarrow \exists !_Z w(t(w) = r(w) = \underline{1})))\end{aligned}$$

As we noted, the restricted quantifiers in some axioms emerge from the fact that they were originally written for sets rather than functions. So, some axioms are not perfectly adequate for functions. However, since we want our axiomatic system to reflect the usual set-theoretic approach, we think the procedure is worthwhile. In order to have a better adjustment to a function-based approach in our axiomatic system, we introduce one last axiom that will be quite useful in our applications.

If $\alpha(x, y)$ is a formula where all the occurrences of x and y are free, then what follows is an axiom:

N12$_\alpha$—Fitness

$$\forall x \exists! y(\alpha(x,y)) \Rightarrow \forall_Z x \exists_Z f \forall t \forall r(x(t) = \underline{1} \Rightarrow (\alpha(t,r) \Rightarrow (f(t) = r)))$$

Roughly speaking, the central idea of this axiom is to connect functions in ZFC with functions in $\mathcal{N}$. We can think of ZFC functions as particular sets of ordered pairs. From an intuitive point of view, the axiom above guarantees the existence of functions f in $\mathcal{N}$ such that, in the scope of the ordered pairs of terms that occur in a formula α, corresponding to a function in ZFC, there is a coincidence between the images by f and the "images" by α. This is important to ensure that ZFC functions can be expressed in $\mathcal{N}$, and so no expressive power is lost when we move to the latter. We will return to this point in Sect. 5, where we'll examine in detail the connection between ZFC and $\mathcal{N}$. Before doing that, let's consider a more intuitive understanding of the theory we propose.

4 Naive Function Theory

For the working mathematician, as well as for the physicist and the philosopher, it will be more interesting to provide an intuitive approach to our set theory, rather than emphasize its formal part. This is one of the purposes of the present section. The other is to highlight key roles that functions play in scientific and mathematical practice.

Our primitive concept is that of a function. Intuitively, a function works as some kind of procedure that assigns objects to other objects; it establishes a mapping between them. Given these mappings, functions have ubiquitous roles in scientific and mathematical theorizing.

- First, applied to linguistic items (e.g., terms in a theory), functions are sometimes used to express relations between the basic concepts of a theory. In particular, the primitive terms of a theory are often connected by suitable functions. One simple example is the function $+: V \times V \rightarrow V$ that corresponds to the addition of vectors in a vector space V. If u and v are vectors (they belong to V), then $+(u, v)$ is another vector.
- Second, applied to propositions (e.g., the axioms of a theory), functions are employed to establish connections between different parts of the same theory. In this way, certain logical relations between components of the theory can be determined and studied. One simple example is the notion of axiom schema. Consider the case of the separation schema **ZF5**$_F$, which establishes a correspondence between any possible formula F of ZF language (with the usual syntactic constraints) and an axiom.
- Third, applied to models, functions are also used to show how different theories are connected to each other. This is accomplished by establishing mappings between certain models of the theories in question. One example is the concept of forgetful functor in category theory. (For details see, for example, Mac Lane 1994.)

- Finally, applied to a mathematical formalism, functions are used to interpret a mathematical theory, assigning a physical interpretation to purely mathematical expressions. Without this central role of functions, most work in applied mathematics and in physics wouldn't be possible. One example is the concept of a physical theory by Günther Ludwig. (For details, see Jenč et al. 1983.)

All the concepts introduced in the formal presentation of $\mathcal{N}$ can be easily examined in the scope of a naive function theory. This theory provides a context to study the functions studied in $\mathcal{N}$. The central feature of functions, as we have already noted, is that they provide mappings connecting objects. In this respect, nothing requires functions to have domains. In fact, the requirement of domains emerges as an artifact of the standard, set-theoretic understanding of functions. In $\mathcal{N}$, an intuitive notion of function, according to which the mapping of objects is the crucial aspect, is preserved.

But, without specifying a domain, what would be for a function to double a real number? The answer, in $\mathcal{N}$, is not a single function—which is as it should be if what counts is the mapping. The answer could be, e.g., a function f such that $f(x) = 2x$ when x is a real number, and $f(x) = x$ when x is not a real number. Another option would be a function g such that $g(x) = 2x$ when x is a real number, and $g(x) = \underline{0}$ otherwise. Since, in $\mathcal{N}$, functions are not associated with domains, a change of domain doesn't affect a function. In fact, strictly speaking, the idea of changing domains is not even part of $\mathcal{N}$—we just have functions.

As a result, we have here a notion of function that captures central, intuitive aspects of this concept. Interestingly, as we'll also see, the notion of function in $\mathcal{N}$ can be easily reduced, in suitable contexts, to the standard, domain-dependent notion. Thus, in the end, what we offer is a genuine generalization of the concept of function, while still preserving its crucial features.

5 Standard Mathematics

Having presented the main features of $\mathcal{N}$, we can now prove that standard mathematics, as formulated in Zermelo–Fraenkel set theory with the axiom of choice (ZFC), is preserved in $\mathcal{N}$. In other words, we prove the following theorem:

Theorem 6 *There is a translation from the language of ZFC into $\mathcal{N}$ such that every translated axiom of ZFC is a theorem in $\mathcal{N}$.*

To prove this theorem, we need to exhibit a translation from ZFC into $\mathcal{N}$. This translation is given by the table below:

Translating ZFC into $\mathcal{N}$

ZFC	$\mathcal{N}$
$\forall$	$\forall_Z$
$\exists$	$\exists_Z$
$\in$	$\in$

Lemma 1 *The translation of* **ZF1** *is a theorem in* $\mathcal{N}$.

Proof The translation of **ZF1** is

$$\forall_Z x \forall_Z y (\forall_Z z (z \in x \Leftrightarrow z \in y) \Rightarrow x = y)$$

which is a straightforward consequence of Definitions 2, 4, and 9 and axiom **N3**. □

Lemma 2 *The translation of* **ZF2** *is a theorem in* $\mathcal{N}$.

Proof The translation of **ZF2** is

$$\exists_Z x \forall_Z y (\neg (y \in x))$$

which is a straightforward consequence of Definitions 2, 4, and 9 and axioms **N2** and **N4**. □

Lemma 3 *The translation of* **ZF3** *is a theorem in* $\mathcal{N}$.

Proof The translation of **ZF3** is

$$\forall_Z x \forall_Z y \exists_Z z \forall_Z t (t \in z \Leftrightarrow t = x \vee t = y)$$

which is a straightforward consequence of Definitions 2, 4, and 9 and axioms **N2** and **N5**. □

Lemma 4 *The translation of* **ZF4** *is a theorem in* $\mathcal{N}$.

Proof The translation of **ZF4** is

$$\forall_Z x \exists_Z y \forall_Z z (z \in y \Leftrightarrow z \subseteq x)$$

which is a straightforward consequence of Definitions 2, 4, and 9 and axioms **N2** and **N6**. □

Lemma 5 *The translation of every axiom* $\mathbf{ZF5}_F$ *is a theorem in* $\mathcal{N}$.

Proof The translation of each axiom $\mathbf{ZF5}_F$ is

$$\forall_Z z \exists_Z y \forall_Z x (x \in y \Leftrightarrow x \in z \wedge F(x))$$

where $F(x)$ is a formula in which there is no free occurrence of the variable y. This is a straightforward consequence of Definitions 2, 4, and 9, and axioms **N2** and $\mathbf{N7}_F$. Observe that $\mathbf{N7}_F$ may be rewritten as

$$\forall_Z f \exists g \forall x (x \in g \Leftrightarrow x \in f \wedge F(x)).$$

There is no need for $\forall_Z x$, since $f(x) = \underline{1}$ $(x \in f)$ entails that x is a ZF-set. It is also unnecessary to write $\exists_Z g$, since we may have the *setification* (see axiom **N2**) of g whenever we want to. □

Lemma 6 *The translation of every axiom* $\mathbf{ZF6}_\alpha$ *is a theorem in* $\mathcal{N}$.

Proof The translation of each axiom $\mathbf{ZF6}_\alpha$ is

$$\forall_Z x \exists_Z ! y \alpha(x,y) \Rightarrow \forall_Z z \exists_Z w \forall_Z t(t \in w \Leftrightarrow \exists_Z s(s \in z \wedge \alpha(s,t)))$$

where $\alpha(x, y)$ is a formula. This is a straightforward consequence of Definitions 2, 4, and 9 and axioms **N2** and **N8**$_\alpha$. □

Lemma 7 *The translation of axiom* **ZF7** *is a theorem in* $\mathcal{N}$.

Proof The translation of axiom **ZF7** is

$$\forall_Z x \exists_Z y \forall_Z z(z \in y \Leftrightarrow \exists_Z t(z \in t \wedge t \in x))$$

which is a straightforward consequence of Definitions 2, 4, and 9 and axioms **N2** and **N9**. □

Lemma 8 *The translation of axiom* **ZF8** *is a theorem in* $\mathcal{N}$.

Proof The translation of axiom **ZF8** is

$$\exists_Z x(\emptyset \in x \wedge \forall_Z y(y \in x \Rightarrow y \cup \{y\} \in x))$$

Now, a straightforward consequence of Definitions 2, 4, and 9 and axioms **N2** and **N10** is the formula:

$$\exists_Z x(\emptyset \in x \wedge \forall_Z y(y \in x \Rightarrow (\forall_Z z(y \subset z \Rightarrow z \in x))))$$

It happens that if $z = y \cup \{y\}$, then $y \subset z$. Hence, the translated axiom is a theorem in $\mathcal{N}$. □

Lemma 9 *The translation of axiom* **ZF9** *is a theorem in* $\mathcal{N}$.

Proof The translation of axiom **ZF9** is

$$\forall_Z x(\forall_Z y \forall_Z z((y \in x \wedge z \in x \wedge y \neq z) \Rightarrow (y \neq \emptyset \wedge y \cap z = \emptyset)) \Rightarrow \exists_Z y \forall_Z z(z \in x \Rightarrow \exists_Z w(y \cap z = \{w\})))$$

which is a straightforward consequence of Definitions 2, 4, and 9 and axioms **N2** and **N11**. □

This last lemma concludes the Proof of Theorem 6. With this theorem in place, we can use standard results from analysis, calculus, linear algebra, geometry, algebra and other branches of mathematics when we reformulate various theories within $\mathcal{N}$.

But before engaging into this reformulation project, something should be said about the axiom **N12**$_\alpha$, the axiom of fitness. Since $\mathcal{N}$'s language encompasses that of ZFC, one could ask: are functions in ZFC translated as functions in $\mathcal{N}$? The answer is obviously negative, since functions in $\mathcal{N}$ have no domains. However, in ZFC, any function is a set of ordered pairs (x, y) with a given domain and a codomain, where both are sets. And, in $\mathcal{N}$, we can define the concept of a ZF-*function*. As a result, we will be able to reproduce in $\mathcal{N}$ the central features that functions have in ZFC.

Definition 10 A *ZF-function* with domain a and codomain b in $\mathcal{N}$ is a set f such that a and b are ZF-sets and

$$\forall x(a(x) = \underline{1} \Rightarrow \exists! y(b(y) = \underline{1} \wedge f(x, y) = \underline{1}))\wedge$$
$$\forall z(f(z) = \underline{1} \Rightarrow \exists x \exists y(a(x) = \underline{1} \wedge b(y) = \underline{1} \wedge z = (x, y)))$$

We are using $f(x, y)$ as an abbreviation for $f((x, y))$. If $f(x, y) = \underline{1}$, we say that y is a *ZF-image* of x by f. In other words, ZF-functions in $\mathcal{N}$ are sets. Recall that, analogously, functions in ZFC are sets as well. So, every ZF-function is a function, but the converse is not necessarily the case.

Recall that axiom $\mathbf{N12}_\alpha$ states that

$$\forall x \exists! y(\alpha(x, y)) \Rightarrow \forall x \exists f \forall t \forall r(x(t) = \underline{1} \Rightarrow (\alpha(t, r) \Rightarrow (f(t) = r)))$$

This means that if an ordered pair (t, r) belongs to a ZF-function g ($g(t, r) = \underline{1}$)—which is guaranteed by the replacement schema—then there is another function f such that $f(t) = r$. This guarantees the existence of functions f (in the sense of $\mathcal{N}$) such that, in the scope of the ordered pairs that belong to a given ZF-function g, there is a coincidence between the images by f and the ZF-images by g. As a result, we can capture in $\mathcal{N}$ the standard, domain-dependent notion of function found in ZFC. In particular, $\mathbf{N12}_\alpha$ guarantees that any function expressible in ZFC corresponds to a function in $\mathcal{N}$. Thus, since there are functions in $\mathcal{N}$ that are not ZF-functions, we have in $\mathcal{N}$ a generalization of the notion of function. And this generalization still preserves, in the case of ZF-functions, the central feature that functions have in ZFC, namely, they are mappings from a domain to a codomain.

Here is a list of some functions that can be expressed in $\mathcal{N}$, with their respective justifications:

- $\exists f \forall x(f(x) = \underline{0})$—from axioms **N4** and **N2**. This function f is a set and it is abbreviated as $\emptyset$.
- $\exists f(f(\emptyset) = \underline{1} \wedge \forall x(x \neq \emptyset \Rightarrow f(x) = \underline{0}))$—from axioms **N6** and **N2**. This function f is also a set and it is abbreviated as $\{\emptyset\}$.
- $\exists f(f(\emptyset) = \underline{1} \wedge f(\{\emptyset\}) = \underline{1} \wedge \forall x((x \neq \emptyset \wedge x \neq \{\emptyset\}) \Rightarrow (f(x) = \underline{0})))$—from axioms **N6** and **N2**. This function f is a set and it is abbreviated as $\{\emptyset, \{\emptyset\}\}$.
- $\exists f(\forall x((x = \emptyset \Rightarrow f(x) = \{\emptyset\}) \wedge (x \neq \emptyset \Rightarrow f(x) = \underline{0})))$—from axioms $\mathbf{N12}_\alpha$ and **N6**. This function f is not a set.

In the next sections, we will show that our formal approach is more economical in the process of axiomatization than the one based on ZFC axioms, by considering the case of significant physical and mathematical theories. Our approach is based on the use of suitable predicates for axiomatization, as introduced by Suppes (2002). But instead of using ZFC as the basic set-theoretic setting, we will use $\mathcal{N}$. In particular, instead of invoking sets, we will employ functions as our primitive notions.

6 A Standard Example in Physics

Our first example of application of $\mathcal{N}$ is an axiomatic framework for a very simple form of non-relativistic classical particle mechanics. The approach is essentially based on the axiomatization of classical particle mechanics due to Suppes (1957), which, in turn, is a variant of the formulation by McKinsey et al. (1953). We call this McKinsey–Sugar–Suppes system of classical particle mechanics "MSS system". A discussion of the MSS system is important because it will allow us to examine the issue of the definability—and, hence, dispensability—of physical concepts, such as set of particles, mass, time and force. For contrast, we will initially present the MSS system in ZFC. We will later reformulate it in $\mathcal{N}$.

It should be pointed out that we don't think that the MSS system faithfully captures every idea behind Newtonian mechanics. But it does seem to capture, in an intuitive manner, at least some of the main aspects of Newton's ideas concerning mechanics.

The MSS system has six primitive notions: P, T, m, $\mathbf{s}$, $\mathbf{f}$, and $\mathbf{g}$. P and T are sets; m is a real-valued unary function defined on P; $\mathbf{s}$ and $\mathbf{g}$ are vector-valued functions defined on the Cartesian product $P \times T$, and $\mathbf{f}$ is a vector-valued function defined on the Cartesian product $P \times P \times T$. Intuitively, P corresponds to the set of particles and T is to be physically interpreted as a set of real numbers measuring elapsed times (in terms of some unit of time, and measured from some origin). In turn, $m(p)$ is to be interpreted as the numerical value of the mass of $p \in P$; whereas $\mathbf{s}_p(t)$, with $t \in T$, is a three-dimensional vector which should be physically interpreted as the position of p at instant t. Moreover, $\mathbf{f}(p, q, t)$, with $p, q \in P$, corresponds to the internal force that the particle q exerts over p at instant t. Finally, the function $\mathbf{g}(p, t)$ is understood as the external force acting on the particle p at instant t.

We can now give the axioms for the MSS system:

Definition 11 $\mathcal{P} = \langle P, T, \mathbf{s}, m, \mathbf{f}, \mathbf{g} \rangle$ is a MSS system if and only if the following axioms are satisfied:

P1 P is a non-empty, finite set.
P2 T is an interval of real numbers.
P3 If $p \in P$ and $t \in T$, then $\mathbf{s}_p(t)$ is a three-dimensional vector ($\mathbf{s}_p(t) \in \Re^3$) such that $\frac{d^2\mathbf{s}_p(t)}{dt^2}$ exists.
P4 If $p \in P$, then $m(p)$ is a positive real number.
P5 If $p, q \in P$ and $t \in T$, then $\mathbf{f}(p, q, t) = -\mathbf{f}(q, p, t)$.
P6 If $p, q \in P$ and $t \in T$, then $[\mathbf{s}_p(t), \mathbf{f}(p, q, t)] = -[\mathbf{s}_q(t), \mathbf{f}(q, p, t)]$.
P7 If $p, q \in P$ and $t \in T$, then $m(p)\frac{d^2\mathbf{s}_p(t)}{dt^2} = \sum_{q \in P} \mathbf{f}(p, q, t) + \mathbf{g}(p, t)$.

Some remarks regarding the axioms are in order here: (a) The brackets in Axiom **P6** denote the external product. (b) Axiom **P5** corresponds to a weak version of Newton's Third Law: to every force there is always a counter-force. (c) Axioms **P6** and **P5** correspond to the strong version of Newton's Third Law. Axiom **P6** establishes that the direction of force and counter-force is the direction of the line defined by the coordinates of particles p and q. (d) Axiom **P7** corresponds to Newton's Second Law.

Now, in the study of a MSS system, it's sometimes useful to consider only certain parts of the system—perhaps only a subsystem needs to be considered. But is the subsystem of a MSS system still a MSS system? In (McKinsey et al. 1953) this question is positively answered in full details. But the point here is not to raise this kind of question. We are interested on the use of Padoa's Principle and its consequences. Now we have all the resources in place to start asking questions regarding the independence of primitive notions in a MSS system. Using Padoa's method, it's not difficult to prove the following theorem:

Theorem 7 *Mass and internal force are each independent of the remaining primitive notions of a MSS system.*

After presenting the MSS system, and in light of the last theorem, Suppes raised a significant issue regarding the definability of the notions of force in the system. As he points out (Suppes 1957):

> Some authors have proposed that we convert the second law [of Newton], that is, **P7**, into a definition of the total force acting on a particle. [...] It prohibits within the axiomatic framework any analysis of the internal and external forces acting on a particle. That is, if all notions of force are eliminated as primitive and **P7** is used as a definition, then the notions of internal and external force are not definable within the given axiomatic framework.

It was natural then to extend Suppes' point even further, by examining the concepts of time and spacetime. And in (da Costa and Sant'Anna 2001, 2002), the authors have proved that time is definable—and, thus, dispensable—in some very natural axiomatic frameworks for classical particle mechanics and even thermodynamics. Furthermore, they have established, in the first paper, that spacetime is also eliminable in general relativity (GR), classical electromagnetism, Hamiltonian mechanics, classical gauge theories, and in the theory of Dirac's electron. Having an axiomatic framework in place allows one to obtain results of this type.

In particular, returning to the MSS system, here is one of the theorems proved in the papers quoted above:

Theorem 8 *Time is eliminable in a MSS system.*

The proof is quite simple. According to Padoa's principle, the primitive concept T in a MSS system is independent from the remaining primitive concepts (mass, position, internal force, and external force) iff there are two models of MSS system such that T has two interpretations and the remaining primitive symbols have the same interpretation. But these two interpretations are not possible, since position $\mathbf{s}$, internal force $\mathbf{f}$, and external force $\mathbf{g}$ are functions whose domains depend on T. If we change the interpretation of T, then we change the interpretation of three other primitive concepts, namely, $\mathbf{s}$, $\mathbf{f}$, and $\mathbf{g}$. So, time is not independent and hence can be defined. Since time is definable, it is eliminable.

In (da Costa and Sant'Anna 2002), the authors have shown that time is dispensable in thermodynamics as well, at least in a particular (although very natural) axiomatic framework for the theory. Moreover, in the same paper, they have shown how to define time and how to restate thermodynamics without any explicit reference to time. In the case of the MSS system, time can be defined by means of the domain of the functions $\mathbf{s}$, $\mathbf{f}$, and $\mathbf{g}$. A similar procedure is used in da Costa and Sant'Anna (2002).

7 Reformulating the MSS System

We can now reformulate the MSS system in the language of $\mathcal{N}$.$\Re^3$ is the usual three-dimensional vector space of ordered triples of real numbers. $\Re^+$ is the usual set of nonnegative real numbers.

Definition 12 $\mathcal{P} = \langle m, \mathbf{s}, \mathbf{f}, \mathbf{g} \rangle$ is an $\mathcal{N}$-MSS system if and only if the following axioms are satisfied:

P1 For all $p, \Re^+(m(p)) = \underline{1}$ and there is at least one function p such that $m(p) > 0$. Moreover, if P is a set such that $P(p) = \underline{1}$ iff $m(p) > 0$, then P is a finite ZF-set.

P2 For all p and for all $t, \Re^3(\mathbf{s}(p,t)) = \underline{1}$ and $\frac{d^2}{dt^2}\mathbf{s}(p,t)$ exists, where $\frac{d^2}{dt^2}\mathbf{s}(p,t)$ is defined as the second derivative of $\mathbf{s}(p,t)$ with respect to t if $\Re(t) = \underline{1}$, and is the null vector if $\Re(t) \neq \underline{1}$. If $m(p) = 0$, then $\mathbf{s}(p, t)$ is a constant vector with respect to t.

P3 For all p, for all q and for all $t, \Re^3(\mathbf{f}(p,q,t)) = \underline{1}$. If either $m(p) = 0$ or $m(q) = 0$, then $\mathbf{f}(p, q, t) = (0, 0, 0)$.

P4 For all p and for all $t, \Re^3(\mathbf{g}(p,t)) = \underline{1}$. If $m(p) = 0$, then $\mathbf{g}(p, t) = (0, 0, 0)$.

P5 For all p, for all q and for all t, $\mathbf{f}(p, q, t) = -\mathbf{f}(q, p, t)$.

P6 For all p, for all q and for all t, $[\mathbf{s}(p, t), \mathbf{f}(p, q, t)] = -[\mathbf{s}(q, t), \mathbf{f}(q, p, t)]$.

P7 For all p, for all q and for all t,

$$m(p)\frac{d^2\mathbf{s}(p,t)}{dt^2} = \sum_{q/m(q)>0} \mathbf{f}(p,q,t) + \mathbf{g}(p,t).$$

This axiomatic system is not exactly equivalent to the MSS system as formulated in ZFC. This emerges from the fact that, in $\mathcal{N}$, functions do not have domains. But an interesting feature emerges from this axiomatization, and it comes very naturally from the emphasis on functions. If p is a function such that $m(p) > 0$, then we say that p is a *particle* (or a *physical particle*). Otherwise, we say that p is a *virtual particle*. Virtual particles, within this system, have zero mass and interact among them with null forces. Furthermore, virtual particles interact with "real" particles with null forces. This feature may sound suspicious at first sight, but we believe that it reflects an important aspect of the practice of physicists. After all, the possibility of picking up a virtual particle and promoting it to the rank of a "real" particle is sometimes useful—for example, in the explanation of perturbations in physical systems.

A classical example is the discovery of Neptune. After Uranus was found, astronomers were having trouble figuring out that planet's orbit. They hypothesized that there was another planet, farther away from Uranus, which was disturbing the orbit of the latter. But Newtonian mechanics alone couldn't determine the precise position of that planet. To help in this task, astronomers used an empirical generalization, called Bode's law. According to this law, the mean distances of the planets from the Sun in astronomical units (AU) follow a pattern. This pattern correctly described, within the range of experimental error, the distances to the Sun of all the seven planets that were known at the time, including Uranus. Now, Bode's law is a *function*, mapping individual planets to their mean distances from the Sun. And by following the pattern generated by this function, astronomers were able to make the required mathematical calculations to determine the position of the new planet. Shortly after that, they were able to observe the planet for the first time—even though the planet turned out to be closer to the Sun than Bode's law predicted (30.07 AU, rather than 38.80 AU). In any case, Neptune was found.

Before its discovery, Neptune obviously wasn't taken to exist. So, following our terminology, Neptune had the behavior of a *virtual particle*, since there was no non-zero mass attributed to it. With the observed perturbations on Uranus' orbit, and the eventual observation of Neptune, the latter was promoted to a particle with finite and non-null mass. At this point, it became a particle within the scope of our axiomatization. Note also that the function that corresponds to Bode's Law does not have a specific domain. It worked even for a planet that had not been discovered yet, namely, Neptune. This episode illustrates both the significant heuristic role of (suitably interpreted) functions, and the importance of virtual particles in scientific research.

A similar (and more recent) phenomenon happened in quantum electrodynamics. It is well known that the zero point energy of the quantum vacuum state cannot be associated with zero energy. There is a finite and non-null energy and even a non-null linear momentum associated with the quantum vacuum. Some physicists associate this residual and unavoidable energy with *virtual photons* that are constantly being created and annihilated. And, once again, this association is made in terms of functions, connecting the virtual photons with the remaining energy. This is another example of a promotion of *virtual particles* (with zero energy and zero linear momentum) to the rank of *physical particles* with finite energy and finite linear momentum. This linear momentum is even measurable and is currently associated with the Casimir effect (for details, see Sant'Anna and de Freitas 2000; Suppes et al. 1996). Nevertheless, we should point out that the usual sense of "virtual particle" in quantum electrodynamics has a different meaning from our virtual particles, since the latter are supposed to have null linear momentum.

Since the emphasis on functions often has a significant heuristic role—such as the one involving virtual particles just mentioned—it's no surprise that physicists present their theories in terms of functions. But functions play additional roles in the foundations of physics and mathematics—as we'll discuss in the next sections.

8 Classical Field Theories

A significant feature of certain axiomatic approaches to physical theories is the unified treatment for these theories that these approaches provide. By exploring suitable predicates for physical theories, it is possible to articulate a unified perspective to accommodate theories as diverse as: Hamiltonian mechanics, classical gauge field theories, Maxwell electromagnetism, the theory of Dirac's electron, and Einstein's GR (for details, see da Costa and Doria 1992). This unified treatment is given by the following axiomatic system:

Definition 13 The species of structures of a *classical physical theory* is given by the 9-tuple:

$$\Sigma = \langle M, G, P, \mathcal{F}, \mathcal{A}, \mathcal{I}, \mathcal{G}, B, \bigtriangledown\varphi = \iota \rangle$$

where

1. M is a finite-dimensional smooth real manifold endowed with a Riemannian metric (associated with spacetime), and G is a finite-dimensional Lie group (associated with transformations among coordinate systems).
2. P is a principal fiber bundle $P(M, G)$ over M with Lie group G.
3. $\mathcal{F}, \mathcal{A}$, and $\mathcal{I}$ are cross-sections of bundles associated with $P(M, G)$, which correspond, respectively, to the field space, potential space, and current or source space.
4. $\mathcal{G} \subseteq Diff(M) \otimes \mathcal{G}'$ is the symmetry group of diffeomorphisms of M, and $\mathcal{G}'$ is the group of gauge transformations of the principal fiber bundle $P(M, G)$.
5. $\bigtriangledown\varphi = \iota$ is a Dirac-like equation, where $\varphi \in \mathcal{F}$ is a field associated with its potential by means of a field equation, and $\iota \in \mathcal{I}$ is a current. This differential equation is subject to the boundary or initial conditions B.

This species of structures provides the mathematical core of a huge variety of classical physical theories, from Hamiltonian mechanics to Einstein's GR. Ultimately, to obtain the particular theories, it's a matter of providing suitable physical interpretations of the relevant mathematical structures. From the dynamical point of view, this can be achieved with specific boundary conditions B.

Some physicists may consider that our suggested interpretation of spacetime as a manifold endowed with a metric does not correspond to the usual way of thinking about spacetime in GR. In this context, spacetime may be thought of as a class of diffeomorphically invariant Riemannian geometries. In this sense, the conceptualization of GR underlying the axiomatic approach is somehow idealized (or, perhaps, naive). In fact, a similar criticism could be made to the axiomatic framework for classical particle mechanics given above (MSS system). After all, forces in classical mechanics are also responsible for deformations (in the case of continuum mechanics), rather than just accelerations. But in the axiomatic approach to classical mechanics above, mass is a constant function. Hence, this approach is not able to accommodate the dynamics of a rocket, which loses fuel during its flight and has a mass loss. Any physical theory admits many possible formulations, and the axiomatic approach just discussed provide some, and certainly not all, of them. Despite the fact that the axiomatization program in physics (in the sense of what mathematicians understand axiomatization nowadays) has existed for many decades, additional developments and refinements are still needed. We chose these examples of physical theories (classical particle mechanics, GR, hamiltonian mechanics, and other field theories) just to illustrate a first comparison between the usual set-theoretic approach and our function-theoretic formal system.

With this axiomatic system for field theories in place, we can now study the issue of the dispensability of various concepts employed in this system. In particular, we obtain the following result:

Theorem 9 *Spacetime M is dispensable in a classical physical theory.*

Proof As we saw, Padoa's principle states that a primitive concept M in Σ is independent from the remaining primitive concepts if and only if there are two models of Σ such that M has two different interpretations, while all the other primitive concepts have the same interpretation. But there aren't two models of Σ satisfying this condition, given that any modification in the interpretation of M entails a modification in the interpretations of the concepts $P, \mathcal{F}, \mathcal{A}, \mathcal{I}, \mathcal{G}$, and B. Therefore, spacetime is not independent and, hence, is definable. This means that spacetime is dispensable. □

Since the Lie group G is used to obtain the principal fiber bundle (G acts freely on P), a similar argument can be employed to prove that G is definable. After all, any modification in the interpretation of G entails a modification in the way that G acts on P, and thus we have a modification in the interpretation of P.

Once again, the underlying set theory used in this axiomatization is ZFC. If we try to rewrite a classical physical theory without any explicit mention of either M or G, the resultant reformulation would be awkward and difficult to understand. What will happen if we adopt $\mathcal{N}$ instead?

9 Reformulating Classical Field Theories

To reformulate classical physical theories in $\mathcal{N}$, we will start by defining a suitable structure. The resulting structure will be similar to the one in Definition 13. However, there won't be any explicit mention of either the spacetime M or the action group G, although, for convenience, they are left implicit in the principal fiber bundle P.

Definition 14 The species of structures of a *classical physical theory* is given by the 7-tuple:

$$\Sigma = \langle P, \mathcal{F}, \mathcal{A}, \mathcal{I}, \mathcal{G}, B, \bigtriangledown\varphi = \iota \rangle$$

where

1. P is a principal fiber bundle $P(M, G)$, when M is a finite-dimensional smooth real manifold endowed with a Riemannian metric, and G is a finite-dimensional Lie group.
2. $\mathcal{F}, \mathcal{A}$, and $\mathcal{I}$ are cross-sections of bundles associated with the principal fiber bundle $P(M, G)$, which correspond, respectively, to the field space, potential space, and current or source space.
3. $\mathcal{G} \subseteq \mathit{Diff}(M) \otimes \mathcal{G}'$ is the symmetry group of diffeomorphisms of the base space M of P, and $\mathcal{G}'$ is the group of gauge transformations of the principal fiber bundle $P(M, G)$.
4. $\bigtriangledown\varphi = \iota$ is a Dirac-like equation, where $\varphi \in \mathcal{F}$ is a field associated with its potential by means of a field equation, and $\iota \in \mathcal{I}$ is a current. This differential equation is subject to the boundary or initial conditions B.

The projection $\pi : P \rightarrow M$ of the principal fiber bundle may be considered to have images $\pi(p)$ equal to $\underline{0}$ if p does not belong to the fiber bundle P.

Consider the case of GR, which is accommodated into our axiomatic system (for details, see da Costa and Doria 1992). GR is often understood as a spacetime theory, since it provides a geometric interpretation of gravitation. However, as we saw, if we understand spacetime as a manifold endowed with a metric, such a manifold is dispensable. So, the explanatory role of spacetime is a little more subtle, even in the limited version of GR presented here.

We take the main goal of GR, from the point of view of the working physicist, to be the development of solutions to Einstein's equations. These equations are a coupled system of ten non-linear differential equations (that are reducible to a Dirac-like equation, according to da Costa and Doria 1992). The equations establish a relation between the energy tensor and a given metric. Both metric and energy tensor are functions. Thus, the main purpose of GR is to establish a particular relation between functions, namely, the energy tensor and the metric. This should be the first point to look for in order to determine the meaning of GR in our formal framework, which we consider to be closer to the practice of physicists than the usual axiomatization by means of Zermelo–Fraenkel set theory. We intend to pursue this point further in future works dealing with the applications of $\mathcal{N}$ theory.

Mathematical physics explores a sophisticated correspondence between mathematics and physics. As the examples of physical theories considered above illustrate, the relational aspect of these theories seems to be clearer when we formulate them in a set theory that considers functions to be more fundamental than sets. As we saw, this was a significant motivation to introduce $\mathcal{N}$.

After considering the case of reformulating physical theories in $\mathcal{N}$, it's worth exploring the changes that $\mathcal{N}$ brings to the axiomatization of mathematical theories. To illustrate the latter point, we will now examine the effect of $\mathcal{N}$ in the formulation of the concept of group.

10 A Mathematical Example: Groups

In the standard approach, a group is an ordered triple $\langle G, *, e\rangle$ that satisfies the following axioms:

G1 G is a non-empty set.
G2 $*$ is a function ($* : G \times G \rightarrow G$), such that if a and b belong to G, then $*(a, b)$ is abbreviated as $a*b$.
G3 $e \in G$.
G4 $\forall a \forall b \forall c(a \in G \wedge b \in G \wedge c \in G \Rightarrow a * (b * c) = (a * b) * c)$.
G5 $\forall a(a \in G \Rightarrow a * e = a)$.
G6 $\forall a(a \in G \Rightarrow \exists b(b \in G \wedge a * b = e))$.

In the language of $\mathcal{N}$, we can rewrite group theory very naturally. In group theory, all we need are two constants: $*$ and e. If $f = (a, b)$, then we abbreviate *(*f*) as $a * b$. Moreover, the sequence of symbols $a * b$ corresponds to the image *(*f*) of a function *f*, which is an ordered pair (a, b).

A group then is an ordered pair $\langle *, e\rangle$ that satisfies the following axioms:

NG1 $\forall a(a * e = a \vee a * e = \underline{0})$.
NG2 $\forall a \forall b \forall c(a * (b * c) = (a * b) * c)$.
NG3 $\forall a(\exists b(a * b = e \vee a * b = \underline{0}))$.

Given a group $\langle *, e\rangle$, if there is a set G such that $\forall a \forall b((G(a) = G(b) = \underline{1}) \Leftrightarrow (a * b \neq \underline{0} \wedge G(a * b) = \underline{1}))$, we say that $\langle G, *, e\rangle$ is a *set group*. Set groups correspond to the groups in ZFC. From the concept of a set group, it is easy to define subgroups, homomorphisms, isomorphisms, finite and infinite groups, and so on. From this point, the whole of group theory can be reformulated.

Mathematicians are more interested in the actions of groups—described by the function *—than in sets of elements of a given group. In other words, what matters to them are the functions rather than the sets. This is the main motivation for this reformulation of group theory.

11 Final Remarks

Standard axiomatizations of physical and mathematical theories in the language of ZFC present several superfluous concepts that are usually introduced as primitive. For example, if functions are understood as set-theoretic constructs, their domains need to be specified. This leads to a commitment to these domains even when the physical theories themselves don't require them. However, if we try to formulate these physical theories in ZFC without referring to the dispensable concepts, the new axiomatizations become quite awkward.

As an alternative framework, we propose a new set theory, inspired in von Neumann's seminal work, but still following certain features of ZFC, given our goal of capturing classical mathematics. In the proposed framework, functions, rather than sets, are assumed as primitive, and the result is the theory $\mathcal{N}$. Based on the examples provided, it seems that the axiomatization of physical and mathematical theories in $\mathcal{N}$'s language is more economical than standard axiomatizations based on ZFC. After all, there's no need to assume superfluous concepts as primitive. Moreover, the emphasis on functions is closer to scientific practice. To conclude this paper, let's spell out these points.

In ZFC, if we change the domain of a function, we change the function itself. In $\mathcal{N}$, functions have no domains at all. Since working mathematicians and physicists usually seem to understand functions as formulas expressing certain mappings instead of sets of ordered pairs, we believe that our approach is more easily identifiable with, and more sensitive to, the common practice of scientists. In mathematics and physics, functions are, of course, extremely important. After all, as we saw, it's in terms of them that scientists ultimately (1) establish the relevant connections between the various concepts introduced in a theory, (2) determine how the different parts of a theory are interrelated, and (3) show how a given theory is connected with other theories in related fields. In contrast, domains and codomains—the required components of the set-theoretic characterization of functions—seem to have at best a technical purpose with minor significance for scientists. The set-theoretic conception forces one to focus not at the most relevant places.

For example, if we understand spacetime as a medium where events happen (similarly to the ether in the late nineteenth century), this spacetime is logically dispensable. In contrast, fields, potentials, and currents are the truly indispensable concepts. These concepts—which are usually described by means of functions—are among the main objects of interest to scientists. Moreover, physical theories are concerned with relations between and among events. And, once again, such relations are usually described in terms of functions. In the end, physics can easily survive without sets. But any reformulation of physics without functions would demand an unimaginable effort.

There's also an additional benefit of adopting the framework put forward here. The framework allows us to preserve central aspects of the traditional set-theoretic approach. This is done in two ways. First, as we saw, the ZFC axioms (suitably translated) are theorems in $\mathcal{N}$, and so virtually all classical mathematics can be captured in our framework—at least the mathematics that can be expressed in ZFC. Second, the framework allows us to preserve even the intuition of those who insist that functions must have domains. All we need to do is qualify how functions act in $\mathcal{N}$. For instance, the sine function is a function f that satisfies the boundary problem:

$$f''(x) + f(x) = 0, \quad f(0) = 0, \quad f'(0) = 1$$

when x is a real number. In the remaining cases, we can consider $f(x)$ as having any arbitrary value without any trouble.

In this respect, following von Neumann, what we offer here is not only an alternative to, but also a generalization of, the classical set-theoretic, ZFC-based approach to the axiomatization of physical and mathematical theories.

This paper should be seen as a first step toward a more comprehensive function-theoretic approach to mathematics and theoretical physics. There is still much to do, of course. For example, given that our approach provides, in some sense, a generalization of the concept of characteristic function, how does it compare with topos theory? Topos theory (Johnstone 1977) is, of course, a very elegant foundational proposal for mathematics that has some similarities with the approach presented in this paper. But the explicit presence of two constants in our framework ($\underline{0}$ and $\underline{1}$) could suggest, at a first sight, that we will not be able to define any notion which is equivalent (by means of a suitable translation) to a subobject classifier in topos theory. In order to address this issue, we need first to establish if it is possible to define categories in $\mathcal{N}$ theory, which we believe we can. Our approach allows us to work with classes, which we refer to as sets in our theory (remember that only ZF-sets work as traditional sets). Since we can define standard functions in our framework, as well as classes of functions, it is conceivable (in principle) that such functions may work as morphisms in a category (class), including monic morphisms (which are essential for the definition of subobject classifiers). In other words, we believe it is possible to develop some kind of algebraic structure which goes far beyond our two constants $\underline{0}$ and $\underline{1}$. If that is possible, we should be able to pursue applications of $\mathcal{N}$ theory in computer science, besides theoretical physics. After all, Cartesian closed categories (which are equivalent to typed lambda calculus) have been a major source of interest to computer scientists (Peirce 1991). But these are tasks for another occasion.

Another interesting area that demands research is homotopy theory. As noted, the so-called sets, in $\mathcal{N}$ theory, work as classes, while ZF-sets work as standard sets. In this context, we should be able to talk about classes of topological objects up to homotopy equivalence. And since all concepts of ZF set theory can be incorporated into our approach (Theorem 6), we can certainly talk about domains and codomains, as it happens in standard homotopy theory (Peirce 1991). We intend to develop these points in future works.

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References

Beth, E. W. (1953). On Padoa's method in the theory of definition. *Indagationes Mathematicæ, 15*, 330–339.

Carnap, R. (1958). *Introduction to logic and its applications*. New York: Dover.

da Costa, N. C. A., & Chuaqui, R. (1988). On Suppes' set theoretical predicates. *Erkenntnis, 29*, 95–112.

da Costa, N. C. A., & Doria, F. A. et al. (1992). Suppes predicates for classical physics. In J. Echeverria (Ed.), *The space of mathematics* (pp. 168–191). Berlin: Walter de Gruyer.

da Costa, N. C. A., & Sant'Anna, A. S. (2001). The mathematical role of time and spacetime in classical physics. *Foundations of Physics Letters, 14*, 553–563.

da Costa, N. C. A., & Sant'Anna, A. S. (2002). Time in thermodynamics. *Foundations of Physics, 32*, 1785–1796.

Goldstein, H. (1980). *Classical mechanics*. Reading: Addison-Wesley.

Haag, R. (1992). *Local quantum physics: fields, particles, algebras*. Berlin: Springer.

Jackson, J. D. (1975). *Classical electrodynamics*. New York: Wiley.

Jenč, F., Maass, W, Melsheimer, O., & Neumann, H, van der Merwe, A. (1983). Gunther Ludwig and the foundations of physics. *Foundations of Physics, 13*, 639–641.

Johnstone, P. T. (1977). *Topos theory*. New York: Academic Press.

Mac Lane, S. (1994). *Categories for the working mathematician*. New York: Springer.

McKinsey, J. C. C., Sugar, A. C., & Suppes, P. (1953). Axiomatic foundations of classical particle mechanics. *Journal of Rational Mechanics and Analysis, 2*, 253–272.

Mendelson, E. (1997). *Introduction to mathematical logic*. London: Chapman & Hall.

Padoa, A. (1900). Essai d'une théorie algébrique des nombres entiers, précédé d'une introduction logique à une théorie déductive quelconque. *Bibliothèque du Congrès International de Philosophie, 3*, 309–365.

Peirce, B. (1991). *Basic category theory for computer scientists*. Cambridge: MIT Press.

Sant'Anna, A. S., & de Freitas, D. C. (2000). The statistical behavior of the quantum vacuum virtual photons in the Casimir effect. *International Journal of Applied Mathematics, 2*, 283–290.

Suppes, P. (1957). *Introduction to logic*. Princeton: van Nostrand.

Suppes, P. (2002). Representation and invariance of scientific structures. CSLI, Stanford.

Suppes, P., Sant'Anna, A.S., & de Barros, J.A. (1996). A particle theory of the Casimir effect. *Foundations of Physics Letters, 9*, 213–223.

Tarski, A. (1983). Some methodological investigations on the definability of concepts. In A. Tarski (Ed.), *Logic, semantics, metamathematics* (pp. 296–319). Indianapolis: Hacket.

von Neumann, J. (1967). An axiomatization of set theory. In J. Heijenoort (Ed.), *From Frege to Gödel* (pp. 346–354). Cambridge: Harvard University Press.